Exercise 2.1

Show that c_0 is a closed subspace of c and c is a closed subspace of l_∞.

(a) c_0 is a closed subspace of c.

Proof: (i) By definition, c_0 is a subset of c.

(ii) For any $\alpha, \beta \in \mathbb{R}$, $x, y \in c_0$,

$$\lim_{k\to\infty} (\alpha x(n) + \beta y(n)) = \alpha \lim_{n\to\infty} x(n) + \beta \lim_{n\to\infty} y(n)$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0$$

Thus, $\alpha x + \beta y \in c_0$

Therefore, c_0 is a subspace of c.

(iii) For any $x_n \in c_0$ ($n \in \mathbb{N}$) and $x \in c$. Suppose $\lim_{n\to\infty} \| x_n - x \|_\infty = 0$.

Fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\| x_{n_0} - x \|_\infty < \varepsilon/2$.

Since $x_{n_0} \in c_0$, there exists $N \in \mathbb{N}$ such that for any $k > N$, $|x_{n_0}(k)| < \varepsilon/2$.

By Triangle Inequality, for any $k > N$,

$$|x(k)| \le |x(k) - x_{n_0}(k)| + |x_{n_0}(k)|$$
$$\le \| x_{n_0} - x \|_\infty + |x_{n_0}(k)|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon.$$

Therefore, $\lim_{k\to\infty} x(k) = 0$.

Hence, c_0 is a closed subspace of c.

(b) c is a closed subspace of l_∞.

Proof: (i) Since any convergent sequence is bounded, c is a subset of l_∞.

(ii) For any $\alpha, \beta \in \mathbb{R}$ and $x, y \in c$,

$$\lim_{k\to\infty}(\alpha x(k)+\beta y(k)) = \alpha \lim_{k\to\infty} x(k) + \beta \lim_{k\to\infty} y(k).$$

Thus, $\alpha x + \beta y \in c$.

Therefore, c is a subspace of l_∞.

(iii) For any $x_n \in c$ $(n \in \mathbb{N})$ and $x \in l_\infty$. Suppose $\lim_{k\to\infty} \|x_n - x\|_\infty = 0$.

Fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\|x_{n_0} - x\|_\infty < \varepsilon/2$.

Since $x_{n_0} \in c$, $L := \lim_{k\to\infty} x_{n_0}(k)$ exists.

Then there exists $N \in \mathbb{N}$ such that for any $k > N$,

$|x_{n_0}(k) - L| < \varepsilon/2$.

By Triangle Inequality, for any $k > N$,

$$\begin{aligned}
|x(k) - L| &\le |x(k) - x_{n_0}(k)| + |x_{n_0}(k) - L| \\
&\le \|x_{n_0} - x\|_\infty + |x_{n_0}(k) - L| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{k\to\infty} x(k) = L$, which implies $x \in c$.

Hence, c is a closed subspace of l_∞.

$\square$

Exercise 2.6

Show that δ_0 is bounded on $C[0,1]$ when equipped with the $\|\cdot\|_\infty$-norm, but not when equipped with the $\|\cdot\|_1$-norm.

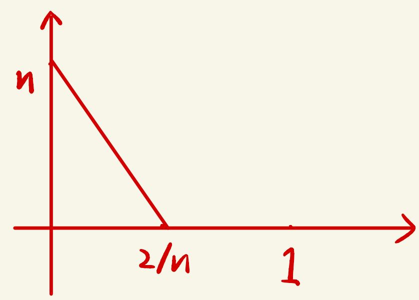
(a) δ_0 is bounded on $(C[0,1], \|\cdot\|_\infty)$.

Proof: For any $f \in C[0,1]$, $|\delta_0 f| = |f(0)| \leq \sup\limits_{x\in[0,1]} |f(x)| = \|f\|_\infty$.

Hence, δ_0 is bounded on $(C[0,1], \|\cdot\|_\infty)$.

(b) δ_0 is not bounded on $(C[0,1], \|\cdot\|_1)$.

Proof: Put $f_n = \begin{cases} -\frac{n^2}{2}x + n, & 0 \leq x \leq \frac{2}{n}, \\ 0, & \text{otherwise}. \end{cases}$ $(n \geq 2)$



Then
$$\|f_n\|_1 = \int_0^1 |f_n(x)|dx = \int_0^{2/n} \left(-\frac{n^2}{2}x + n\right)dx = 1.$$

On the other hand, $|\delta_0(f_n)| = |f_n(0)| = n \to \infty$ as $n \to \infty$.

Hence, δ_0 is not bounded on $(C[0,1], \|\cdot\|_1)$.

$\square$

Exercise 2.9

Prove that c_0 is not a Banach space in the $\|\cdot\|_2$ - norm.

Proof: Put $x(k) = \frac{1}{\sqrt{k}}$.

Since $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, $x \in c_0$.

However, $\|x\|_2 = \left(\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}}\right)^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} \frac{1}{k} \right)^{\frac{1}{2}} = \infty$.

Hence, $\|\cdot\|_2$ is not a norm on c_0.

$\square$